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Bohr radius for subordinating families of analytic functions and bounded harmonic mappings

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ABSTRACT

The class consisting of analytic functions f in the unit disk satisfying $f + \alpha z f' + \gamma z^2 f''$ subordinated to some function h is considered. The Bohr radius for this class is obtained when h is respectively convex or starlike. The Bohr radius for analytic functions mapping the unit disk into a concave-wedge domain as well as for bounded harmonic mappings are also established.

1. Introduction

The Bohr inequality describes the size of the sum of the moduli of the terms in the series expansion of a bounded analytic function. Specifically it states that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in the unit disk $U := \{z : |z| < 1\}$ and $|f(z)| < 1$ for all $z \in U$, then $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ for all $|z| \leq 1/3$. This inequality was obtained by Bohr [10] in 1914, and the constant $r_0 = 1/3$ is known as the Bohr radius. Bohr actually obtained the inequality for $|z| \leq 1/6$, but subsequently later, Wiener, Riesz and Schur, independently established the sharp inequality for $|z| \leq 1/3$ [16,23,25]. Other proofs have also been given in [17–19].

More generally, the Bohr radius for bounded analytic functions in the unit disk can be paraphrased in terms of its supremum norm, that is, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $\|f\|_\infty = \sup_{|z|<1} |f(z)| < \infty$, then

$$\sum_{n=0}^{\infty} \left|a_n z^n\right| \leq \|f\|_\infty$$

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when $|z| \leq 1/3$. Boas and Khavinson [9], and Aizenberg [3,4,6] have extended the inequality to several complex variables. More recently Defant et al. [11] obtained the optimal asymptotic estimate for the n-dimensional Bohr radius on the polydisk U^n.

Operator algebraists have also taken a keen interest in the Bohr inequality, particularly after Dixon [12] used it to settle in the negative a conjecture on Banach algebras. Pursuant to this construction, Paulsen and Singh [18] have extended the Bohr inequality in the context of Banach algebras.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the Bohr inequality can be put in the form

$$d\left(\sum_{n=0}^{\infty} |a_n z^n|, |a_0|\right) = \sum_{n=1}^{\infty} |a_n z^n| \leq d\big(f(0), \partial U\big), \tag{1}$$

where d is the Euclidean distance. More generally, a class of analytic (or harmonic) functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ mapping U into a domain Ω is said to satisfy a Bohr phenomenon if an inequality of type (1) holds uniformly in $|z| < \rho_0$, $0 < \rho_0 \leq 1$, and for all functions in the class. The notion of the Bohr phenomenon was first introduced in [8] for a Banach space X of analytic functions in the disk U. It was shown that under the usual norm, the Bohr phenomenon does not hold for the Hardy spaces H^p, $1 \leq p < \infty$. However use of a different norm might lead to the occurrence of a Bohr phenomenon. In [8], a characterization of appropriate norms was obtained that yielded the Bohr phenomenon for X.

An important notion in complex function theory is subordination. Given two analytic functions f and g, the function g is subordinate to f, written $g(z) \prec f(z)$, if g is the composition of f with an analytic self-map w of the unit disk with $w(0) = 0$. In the case f is univalent, subordination is equivalent to $g(U) \subset f(U)$ and $g(0) = f(0)$. For additional details on subordination classes, see for example [13, Chapter 6] or [20, p. 35].

To make precise the notion of the Bohr phenomenon for classes of functions, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a given analytic function in U with $f(U) = \Omega$. Denote by $S(f)$ the class of analytic functions g subordinate to f. The class $S(f)$ is said to satisfy a Bohr phenomenon if there is a constant $\rho_0 \in (0, 1]$ satisfying

$$\sum_{n=1}^{\infty} |b_n z^n| \leq d\big(f(0), \partial\Omega\big)$$

for all $|z| < \rho_0$, and for any $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$. The constant ρ_0 is called the Bohr radius.

When f is convex, that is, $f(U)$ is a convex domain, Aizenberg [5, Theorem 2.1] showed that the Bohr radius for $S(f)$ is $\rho_0 = 1/3$, a result which includes (1) when $\Omega = U$. Abu-Muhanna [1, Theorem 1] showed that $S(f)$ has a Bohr phenomenon for f univalent, and that the sharp Bohr radius is $3 - 2\sqrt{2} \cong 0.17157$. Equality is attained for the Koebe function $f(z) = z/(1-z)^2$. In a recent paper [15], we had studied the Bohr phenomenon for functions mapping the unit disk into the exterior of a compact convex set.

In Section 2, the Bohr radius is obtained for the class of analytic functions mapping U into a concave-wedge domain. This result established a link between the results of Aizenberg [5] and Abu-Muhanna [1]. Section 3 deals with subordinating families to convex or starlike functions. Specifically the class $R(\alpha, \gamma, h)$ consisting of analytic functions f satisfying $f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z)$ in U is considered. The Bohr radius is obtained for $R(\alpha, \gamma, h)$ when h is respectively convex or starlike. The final section is devoted to finding the Bohr radius for bounded harmonic mappings in the unit disk. Connections of the results obtained in this paper to several earlier works will also be illustrated.

2. Bohr's radius for concave-wedge domains

A link to the earlier results of Aizenberg [5] and Abu-Muhanna [1] could be established by considering the concave-wedge domains

$$W_\alpha := \left\{ w \in \mathbb{C} : |\arg w| < \frac{\alpha\pi}{2} \right\}, \quad 1 \leq \alpha \leq 2. \tag{2}$$

In this instance, the conformal map of U onto W_α is given by

$$F_{\alpha,t}(z) = t\left(\frac{1+z}{1-z}\right)^\alpha = t\left(1+\sum_{n=1}^\infty A_n z^n\right), \quad t>0. \tag{3}$$

When $\alpha = 1$, the domain reduces to a convex half-plane, while the case $\alpha = 2$ yields a slit domain. Denote by S_{W_α} the class consisting of analytic functions f mapping the unit disk U into the wedge domain W_α given by (2).

The following result of [2] will be needed.

Proposition 2.1. *If F is an analytic univalent function mapping U onto Ω, where the complement of Ω is convex and $F(z) \neq 0$, then any analytic function $f \in S(F^n)$, $n = 1, 2, \ldots$, can be expressed as*

$$f(z) = \int\limits_{|x|=1} F^n(xz)\, d\mu(x)$$

for some probability measure μ on the unit circle $|x| = 1$. Consequently,

$$f(z) = \int\limits_{|x|=1} \exp\bigl(F(xz)\bigr)\, d\mu(x)$$

for every $f \in S(\exp(F))$.

The following result will also be helpful.

Lemma 2.2. *Let $F_{\alpha,t}(z) = t((1+z)/(1-z))^\alpha = t(1+\sum_{n=1}^\infty A_n z^n)$ be given by (3), $\alpha \in [1,2]$. Then $A_n > 0$ for all n.*

Proof. Evidently

$$F'_{\alpha,t}(z) = \frac{2\alpha}{1-z^2} F_{\alpha,t}(z). \tag{4}$$

Expanding (4) leads to

$$\sum_{n=1}^\infty n A_n z^{n-1} = 2\alpha\left(1+\sum_{n=1}^\infty z^{2n}\right)\left(1+\sum_{n=1}^\infty A_n z^n\right),$$

and thus

$$A_{n+1} = \frac{2\alpha}{n+1}\sum_{k=0}^{[\frac{n}{2}]} A_{n-2k} \tag{5}$$

for all $n \geq 1$, where $[\ ]$ is the greatest integer function and $A_0 = 1$.

It follows by induction that

$$A_n = p_n(\alpha) \tag{6}$$

is a polynomial of degree n with positive coefficients. Indeed it holds for $n = 1$ since $A_1 = 2\alpha > 0$. Assuming that (6) holds for $n = m$, then (5) yields

$$A_{m+1} = \frac{2\alpha}{m+1}\sum_{k=0}^{[\frac{m}{2}]} A_{m-2k} = \frac{2\alpha}{m+1}\sum_{k=0}^{[\frac{m}{2}]} p_{m-2k}(\alpha) = p_{m+1}(\alpha)$$

where

$$p_{m+1}(\alpha) = \frac{2\alpha}{m+1}\sum_{k=0}^{[\frac{m}{2}]} p_{m-2k}(\alpha).$$

Since p_{m+1} is a polynomial of degree $m+1$ with positive coefficient in each term, evidently (6) is true for all $n \geq 1$. Consequently, $A_n = p_n(\alpha) > 0$ for all $n \geq 1$. $\square$

The following are the main results for this section.

Theorem 2.3. *Let $\alpha \in [1, 2]$. If $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in S_{W_\alpha}$ with $a_0 > 0$, then*

$$\sum_{n=1}^{\infty} \left|a_n z^n\right| \leq d(a_0, \partial W_\alpha)$$

for $|z| \leq r_\alpha = (2^{1/\alpha} - 1)/(2^{1/\alpha} + 1)$. The function $f = F_{\alpha,a_0}$ in (3) shows that the Bohr radius r_α is sharp.

Proof. Write

$$f^*(z) = \sum_{n=0}^{\infty} |a_n| z^n.$$

Since $f \in S(F_{\alpha,a_0})$, it follows from Proposition 2.1 and Lemma 2.2 that

$$\begin{aligned} f^*(r) - a_0 &\leq a_0 \sum_{n=1}^{\infty} A_n r^n = a_0 \left[\left(\frac{1+r}{1-r}\right)^\alpha - 1\right] \\ &= d(a_0, \partial W_\alpha)\left[\left(\frac{1+r}{1-r}\right)^\alpha - 1\right] \leq d(a_0, \partial W_\alpha) \end{aligned}$$

for $|z| = r \leq r_\alpha$, where r_α is the smallest positive root of the equation

$$\left(\frac{1+r}{1-r}\right)^\alpha - 1 = 1.$$

Thus $r_\alpha = (2^{\frac{1}{\alpha}} - 1)/(2^{\frac{1}{\alpha}} + 1)$. $\square$

Theorem 2.4. *Let $\alpha \in [1, 2]$. If $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in S_{W_\alpha}$, then*

$$\sum_{n=0}^{\infty} \left|a_n z^n\right| - |a_0|^* \leq d\big(|a_0|^*, \partial W_\alpha\big)$$

for $|z| \leq r_\alpha = (2^{1/\alpha} - 1)/(2^{1/\alpha} + 1)$, where $|a_0|^* = F_{\alpha,1}(|F_{\alpha,1}^{-1}(a_0)|)$ and $F_{\alpha,1}$ is given by (3). The function $f = F_{\alpha,|a_0|^*}$ shows that the Bohr radius r_α is sharp.

Proof. Since $f \in S_{W_\alpha}$, there exists $b \in U$ such that $F_{\alpha,1}(b) = a_0$. Let

$$\varphi(z) = \sum_{n=0}^{\infty} b_n z^n = \frac{z+b}{1+\bar{b}z} = b + \big(1 - |b|^2\big)\sum_{n=1}^{\infty} (-\bar{b})^{n-1} z^n.$$

Then $(F_{\alpha,1} \circ \varphi)(0) = F_{\alpha,1}(\varphi(0)) = a_0 = f(0)$ which yields

$$f \prec F_{\alpha,1} \circ \varphi. \tag{7}$$

Next, let $|a_0|^* = F_{\alpha,1}(|b|)$. Then

$$|a_0|^* = F_{\alpha,1}\big(\big|F_{\alpha,1}^{-1}(a_0)\big|\big) \geq \big|F_{\alpha,1}\big(F_{\alpha,1}^{-1}(a_0)\big)\big| = |a_0|.$$

Now the function

$$\varphi^*(z) = \sum_{n=0}^{\infty} |b_n| z^n = \frac{|b| + (1 - 2|b|^2)z}{1 - |b|z}$$

maps the disk $|z| < 1/3$ into U. Thus $G^*(z) = F_{\alpha,1}(\varphi^*(z))$ satisfies

$$G^* \prec F_{\alpha,|a_0|^*} \tag{8}$$

for $|z| < 1/3$. Further for any analytic functions g_1, g_2 defined on U,

$$(g_1 + g_2)^*\big(|z|\big) \leq g_1^*\big(|z|\big) + g_2^*\big(|z|\big) \quad \text{and} \quad (g_1 g_2)^*\big(|z|\big) \leq g_1^*\big(|z|\big) g_2^*\big(|z|\big)$$

which give

$$(F_{\alpha,1} \circ \varphi)^*(r) \leq G^*(r). \tag{9}$$

Hence using (7), (8) and (9) together with Proposition 2.1, it follows that

$$f^*(r) \leq (F_{\alpha,1} \circ \varphi)^*(r) \leq G^*(r) \leq F_{\alpha,|a_0|^*}(r) = |a_0|^* \left(\frac{1+r}{1-r}\right)^{\alpha}$$

for $r \leq 1/3$. Consequently, $f^*(r) - |a_0|^* \leq |a_0|^* = d(|a_0|^*, \partial W_\alpha)$ provided $r \leq r_\alpha$, where r_α is the smallest positive root of

$$\left(\frac{1+r}{1-r}\right)^{\alpha} - 1 = 1,$$

that is, $r_\alpha = (2^{\frac{1}{\alpha}} - 1)/(2^{\frac{1}{\alpha}} + 1)$. $\square$

Remark 2.5. Since $\alpha \in [1, 2]$, it follows that $0.17157 \approx (\sqrt{2} - 1)/(\sqrt{2} + 1) \leq r_\alpha \leq \frac{1}{3}$.

Remark 2.6. If $a_0 \geq 1$, then $|a_0|^* = a_0$ and Theorem 2.4 is equivalent to Theorem 2.3. However the case $0 < a_0 < 1$ gives $|a_0|^* = 1/a_0$.

Remark 2.7. The Bohr radius for the half-plane is $r_1 = 1/3$, and $r_2 = 3 - 2\sqrt{2}$ for the slit-map. Since every convex domain lies in a half-plane, it readily follows from Theorem 2.3 that the Bohr radius for convex domains is $1/3$. When the class of functions is subordinate to an analytic univalent function, it follows from de Brange's Theorem [14] that the moduli of its Taylor's coefficients are bounded by the coefficients of the slit-map, which from Theorem 2.3, readily yields the Bohr radius $3 - 2\sqrt{2}$ for this class [1].

3. Second-order differential subordination

For $\alpha \geq \gamma \geq 0$, and for a given analytic convex function $h \in \mathcal{A}$, let

$$R(\alpha, \gamma, h) := \left\{ f \in \mathcal{A} : f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z),\ z \in U \right\}.$$

The investigation of such functions f can be seen as an extension to the study of the class

$$R(\alpha, h) = \left\{ f \in \mathcal{A} : f'(z) + \alpha z f''(z) \prec h(z),\ z \in U \right\}$$

or its variations for an appropriate function h. This class has been investigated in several works, and more recently in [24,26]. It was shown in Ali et al. [7] that $f(z) \prec h(z)$ whenever $f \in R(\alpha, \gamma, h)$. The notion of convolution will be needed to deduce the latter assertion.

Denote by $\mathcal{A}$ the class of all analytic functions f in U. For two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $\mathcal{A}$, the Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The following auxiliary function will be useful: let

$$\phi_\lambda(z) = \int_0^1 \frac{dt}{1 - zt^\lambda} = \sum_{n=0}^{\infty} \frac{z^n}{1 + \lambda n}.$$

From [21] it is known that ϕ_λ is convex in U provided $\operatorname{Re} \lambda \geq 0$.

Now for $\alpha \geq \gamma \geq 0$, let

$$\nu + \mu = \alpha - \gamma, \quad \mu\nu = \gamma,$$

and

$$q(z) = \int_0^1 \int_0^1 h\left(zt^\mu s^\nu\right) dt ds = (\phi_\nu * \phi_\mu) * h(z). \tag{10}$$

Since q is the convolution of convex maps, q itself is convex [22]. It is also easily verified that $q \in R(\alpha, \gamma, h)$. In [7], Ali et al. showed that

$$f(z) \prec q(z) \prec h(z)$$

for every $f \in R(\alpha, \gamma, h)$. Thus $R(\alpha, \gamma, h) \subset S(h)$. The following result gives the best Bohr radius for $R(\alpha, \gamma, h)$.

Theorem 3.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in R(\alpha, \gamma, h)$, and h be convex. Then*

$$\sum_{n=1}^{\infty} \left|a_n z^n\right| \leq d\left(h(0), \partial h(U)\right)$$

for all $|z| \leq r_{CV}(\alpha, \gamma)$, where $r_{CV}(\alpha, \gamma)$ is the smallest positive root of the equation

$$(\phi_\mu * \phi_\nu)(r) - 1 = \sum_{n=1}^{\infty} \frac{1}{(1 + \mu n)(1 + \nu n)} r^n = \frac{1}{2}.$$

Further, this bound is sharp. An extremal case occurs when $f(z) := q(z)$ as defined in (10) and $h(z) := l(z) = \frac{z}{1-z}$.

Proof. Let $F(z) = f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z)$. Then

$$F(z) = \sum_{n=0}^{\infty} \bigl[1 + \alpha n + \gamma n(n-1)\bigr] a_n z^n,$$

and

$$\frac{1}{h'(0)} \sum_{n=1}^{\infty} \bigl[1 + \alpha n + \gamma n(n-1)\bigr] a_n z^n = \frac{F(z) - F(0)}{h'(0)} \prec \frac{h(z) - h(0)}{h'(0)}.$$

It follows from [13, Theorem 6.4(i)] that

$$\left| \frac{1 + \alpha n + \gamma n(n-1)}{h'(0)} \right| |a_n| \le 1, \quad n \ge 1.$$

Hence

$$|a_n| \le \frac{|h'(0)|}{1 + (\mu + \nu)n + \mu\nu n^2}, \quad n \ge 1,$$

which readily yields

$$\sum_{n=1}^{\infty} |a_n| r^n \le \sum_{n=1}^{\infty} \frac{|h'(0)|}{1 + (\mu + \nu)n + \mu\nu n^2} r^n.$$

Since $H(z) = \frac{h(z)-h(0)}{h'(0)}$ is a normalized convex function in U, it follows that

$$d(0, \partial\Omega) \ge 1/2, \quad \Omega = H(U),$$

implying

$$d\bigl(h(0), \partial h(U)\bigr) = \inf_{\zeta \in \partial U} \bigl|h(\zeta) - h(0)\bigr| \ge \frac{|h'(0)|}{2}, \quad z \in U.$$

Thus

$$\sum_{n=1}^{\infty} |a_n| r^n \le 2d\bigl(h(0), \partial h(U)\bigr) \left(\sum_{n=1}^{\infty} \frac{1}{(1 + \mu n)(1 + \nu n)} r^n \right),$$

and the Bohr radius $r_{CV}(\alpha, \gamma)$ is the smallest positive root of the equation

$$\sum_{n=1}^{\infty} \frac{1}{(1 + \mu n)(1 + \nu n)} r^n = \frac{1}{2}. \qquad \square \tag{11}$$

The accompanying graph in Fig. 1 describes the extremal case. With $h(z) := l(z) = z/(1 - z)$, then $d(h(0), \partial h(U)) = 1/2$, and $q(z) = (\phi_\mu * \phi_\nu) * l(z)$ maps the Bohr circle of radius r_{CV} into $\{w : |w| \le 1/2\}$. Here the image of the Bohr circle is depicted by a bold closed curve.

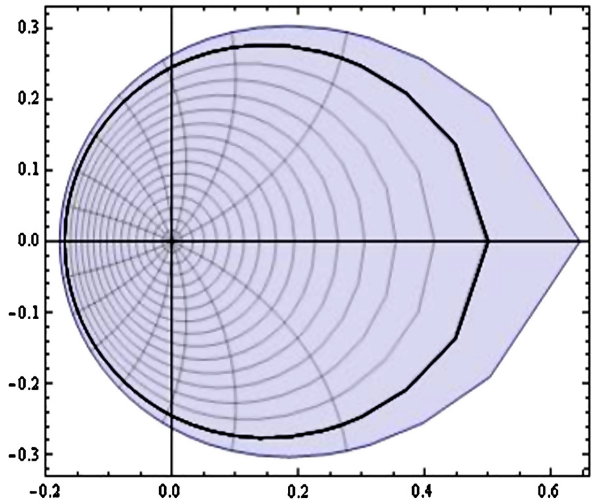

Fig. 1. Image of the Bohr circle under $q(z) = (\phi_\mu * \phi_\nu) * \frac{z}{1-z}$ for $\alpha = 3$, $\gamma = 1$.

Table 1
The Bohr radius $r_{CV}(\alpha, \gamma)$ for different α and γ.

α	$r_{CV}(\alpha, 0)$
0	0.333333
0.1	0.365245
1	0.582812
10	0.994200
20	0.999958
28	0.999999

α	γ	$r_{CV}(\alpha, \gamma)$
0	0	0.333333
1	0.5	0.649755
1	0.9	0.684027
4	0.9	0.981325
4	1	0.986793
4	4/3	0.999999

Remark 3.2. The Bohr radius $r_{CV}(0, 0) = 1/3$ was obtained in [5, Theorem 2.1].

From (11), it is known that for any $f \in R(\alpha, \gamma, h)$ and h convex, the Bohr radius $r_{CV}(\alpha, \gamma)$ can be found by solving the equation

$$\sum_{n=1}^{\infty} \frac{1}{1 + \alpha n + \gamma n(n-1)} r^n = \sum_{n=1}^{\infty} \frac{1}{(1 + \mu n)(1 + \nu n)} r^n = \frac{1}{2}$$

for the smallest positive root. Table 1 gives the values of the Bohr radius for different choices of the parameters α and γ. Note that $r_{CV}(\alpha, \gamma)$ approaches 1 for increasing α and γ.

The following theorem deals with subordination to a starlike function.

Theorem 3.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in R(\alpha, \gamma, h)$, and h be starlike. Then*

$$\sum_{n=1}^{\infty} |a_n z^n| \le d\bigl(h(0), \partial h(U)\bigr)$$

for all $|z| \le r_{ST}(\alpha, \gamma)$, where $r_{ST}(\alpha, \gamma)$ is the smallest positive root of the equation

$$(\phi_\mu * \phi_\nu)(r) - 1 = \sum_{n=1}^{\infty} \frac{n}{(1 + \mu n)(1 + \nu n)} r^n = \frac{1}{4}.$$

This bound is sharp. An extremal case occurs when $f(z) := q(z)$ as defined in (10) and $h(z) := k(z) = \frac{z}{(1-z)^2}$.

Proof. Since

$$F(z) = f(z) + \alpha z f'(z) + \gamma z^2 f''(z) = \sum_{n=0}^{\infty} \bigl[1 + \alpha n + \gamma n(n-1)\bigr] a_n z^n \prec h(z),$$

it follows that

$$\frac{1}{h'(0)} \sum_{n=1}^{\infty} \bigl[1 + \alpha n + \gamma n(n-1)\bigr] a_n z^n = \frac{F(z) - F(0)}{h'(0)} \prec \frac{h(z) - h(0)}{h'(0)} = H(z).$$

Thus [13, Theorem 6.4(ii)]

$$\left| \frac{1 + \alpha n + \gamma n(n-1)}{h'(0)} \right| |a_n| \le n, \quad n \ge 1,$$

which yields

$$\sum_{n=1}^{\infty} |a_n| r^n \le \sum_{n=1}^{\infty} \frac{n |h'(0)|}{1 + (\mu + \nu)n + \mu\nu n^2} r^n.$$

Since $H(z)$ is a normalized starlike function in U, then

$$\bigl|H(z)\bigr| \ge 1/4, \quad z \in \partial U,$$

implying

$$d\bigl(h(0), \partial h(U)\bigr) = \inf_{\zeta \in \partial U} \bigl|h(\zeta) - h(0)\bigr| \ge \frac{|h'(0)|}{4}, \quad z \in U.$$

Thus

$$\sum_{n=1}^{\infty} |a_n| r^n \le 4d\bigl(h(0), \partial h(U)\bigr) \left(\sum_{n=1}^{\infty} \frac{n}{(1 + \mu n)(1 + \nu n)} r^n \right),$$

and the Bohr radius $r_{ST}(\alpha, \gamma)$ is the smallest positive root of the equation

$$\sum_{n=1}^{\infty} \frac{n}{(1 + \mu n)(1 + \nu n)} r^n = \frac{1}{4}. \qquad \square \tag{12}$$

If h is starlike, then q as given in (10) is starlike. Fig. 2 describes an extremal case. Here $d(h(0), \partial h(U)) = 1/4$ for $h(z) := k(z) = z/(1-z)^2$, and $q(z) = (\phi_\mu * \phi_\nu) * k(z)$ maps the Bohr circle, depicted as the bold closed curve, into $\{w : |w| \le 1/4\}$.

Remark 3.4. The Bohr radius $r_{ST}(0, 0) = 3 - 2\sqrt{2}$ is equal to the Bohr radius for the class of analytic functions subordinated to a univalent function, see [1, Theorem 1].

From (12), the Bohr radius r_{ST} can be found by solving the equation

$$\sum_{n=1}^{\infty} \frac{n}{1 + \alpha n + \gamma n(n-1)} r^n = \sum_{n=1}^{\infty} \frac{n}{(1 + \mu n)(1 + \nu n)} r^n = \frac{1}{4}$$

for a positive real root. Several values of $r_{ST}(\alpha, \gamma)$ are listed in Table 2.

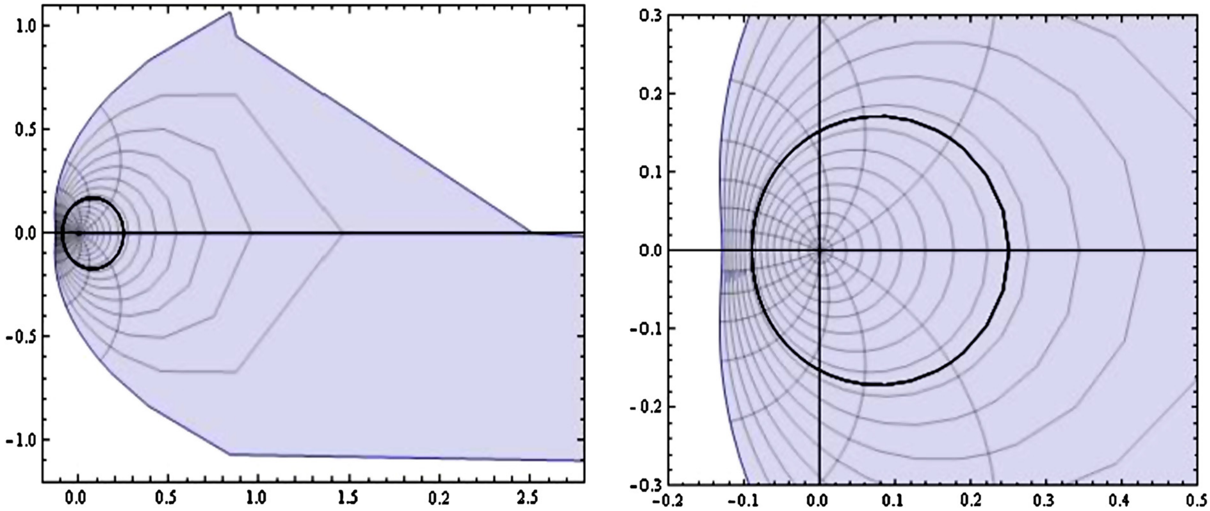

Fig. 2. Image of the Bohr circle under $q(z) = (\phi_\mu * \phi_\nu) * \frac{z}{(1-z)^2}$ for $\alpha = 3$, $\gamma = 1$.

Table 2
The Bohr radius $r_{ST}(\alpha, \gamma)$ for various α and γ.

α	$r_0(\alpha, 0)$
0	0.171573
0.1	0.188154
1	0.308210
10	0.723763
100	0.961586
1 000 000	0.999996

α	γ	$r_0(\alpha, \gamma)$
0	0	0.171573
1	0.1	0.315797
2	1	0.459619
10	1	0.765923
100	10	0.994215
100	35	0.999963

4. Bohr radius for bounded harmonic functions

We conclude by finding the Bohr radius for bounded harmonic functions in the disk. Let D be a bounded set and denote by $\overline{D}$ the closure of D. Let $\overline{D}_{min}$ be the smallest closed disk containing the closure of D. Thus

$$\overline{D} \subseteq \overline{D}_{min} \subseteq \overline{E}$$

for any closed disk $\overline{E}$ containing $\overline{D}$. The following two lemmas are required to deduce the main theorem in this section.

Lemma 4.1. *(See [1].) If $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$, and $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is convex with $f(U) = \Omega$, then*

$$|b_n| \le |a_1| \le 2d\big(f(0), \partial\Omega\big).$$

Lemma 4.2. *Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}$ be a complex-valued harmonic function in U. If f maps U into a bounded domain D, then*

$$\left|e^{i\mu} a_n + e^{-i\mu} b_n\right| \le 2\big(\rho - \left|\operatorname{Re} e^{i\mu}(a_0 - w_0)\right|\big), \tag{13}$$

$$|a_n| + |b_n| \le \frac{4}{\pi}\rho, \tag{14}$$

for any real μ and any $n \ge 1$, where ρ and w_0 are respectively the radius and center of $\overline{D}_{min}$.

Proof. Now $f(U)$ is contained in a disk with radius ρ and center w_0, and so

$$\rho = \left|f(z) - w_0\right| + d\left(f(z), \partial\overline{D}_{min}\right), \quad \text{or} \quad \left|f(z) - w_0\right| = \rho - d\left(f(z), \partial\overline{D}_{min}\right);$$

that is,

$$\left|f(z) - w_0\right| \leq \rho$$

for all $z \in U$. Consequently

$$\left|\operatorname{Re}\left(e^{i\mu}\left[f(z) - w_0\right]\right)\right| \leq \left|e^{i\mu}\left[f(z) - w_0\right]\right| \leq \rho$$

for any real μ and any $z \in D$.

Let

$$W_\mu(z) = e^{i\mu}\left(h(z) - w_0\right) + e^{-i\mu} g(z).$$

Then W_μ is analytic, $W_\mu(0) = e^{i\mu}(a_0 - w_0)$ and $|\operatorname{Re} W_\mu(z)| = |\operatorname{Re}(e^{i\mu}[f(z) - w_0])| < \rho$.

The function

$$F(z) = \frac{2i}{\pi} \rho \log \frac{1+z}{1-z}$$

maps U conformally onto the strip $P = \{\zeta : |\operatorname{Re}\zeta| < \rho\}$. As $e^{i\mu}(a_0 - w_0) \in P$, choose $b \in U$ so that $F(b) = e^{i\mu}(a_0 - w_0)$ and let

$$\varphi(z) = \frac{z+b}{1+\bar{b}z}.$$

Then $F(\varphi(0)) = e^{i\mu}(a_0 - w_0) = W_\mu(0)$, and hence W_μ is subordinate to $F \circ \varphi$.

Simple calculations give

$$(F \circ \varphi)'(0) = \frac{4i(1 - |b|^2)}{\pi(1 - b^2)} \rho \quad \text{and so} \quad \left|(F \circ \varphi)'(0)\right| \leq \frac{4}{\pi} \rho. \tag{15}$$

As $F \circ \varphi$ is convex and

$$d\left(F\left(\varphi(0)\right), \partial P\right) = \rho - \left|\operatorname{Re} W_\mu(0)\right| = \rho - \left|\operatorname{Re} e^{i\mu}(a_0 - w_0)\right|,$$

Lemma 4.1 implies (13). In addition, Lemma 4.1 and (15) imply

$$\left|e^{i\mu} a_n + e^{-i\mu} b_n\right| \leq \frac{4}{\pi} \rho.$$

If $a_n = 0$, then inequality (14) is evident. If $a_n \neq 0$, then

$$\left|e^{i\mu} a_n + e^{-i\mu} b_n\right| = |a_n| \left|1 + e^{-2i\mu} \left(\frac{b_n}{a_n}\right)\right|,$$

and μ can be chosen so that $e^{-2i\mu}(b_n/a_n) = |b_n/a_n|$, which gives (14). □

Remark 4.3. When $D = U$, Lemma 4.2 reduces to Lemma 4 in [1].

The following is our main result in this section.

Theorem 4.4. *Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}$ be a complex-valued harmonic function in U. If $f : U \to D$ for some bounded domain D, then, for $|z| \le 1/3$,*

$$\sum_{n=1}^{\infty} \left|a_n z^n\right| + \sum_{n=1}^{\infty} \left|b_n z^n\right| \le \frac{2}{\pi}\rho \tag{16}$$

and

$$\sum_{n=1}^{\infty} \left|e^{i\mu} a_n + e^{-i\mu} b_n\right| |z|^n + \left|\operatorname{Re} e^{i\mu}(a_0 - w_0)\right| \le \rho, \tag{17}$$

where ρ and w_0 are respectively the radius and center of $\overline{D}_{min}$.

The bound $1/3$ is sharp as demonstrated by an analytic univalent mapping f from U onto D. In particular, if D is an open disk with radius $\rho > 0$ centered at ρw_0, then sharpness is shown by the Möbius transformation

$$\varphi(z) = e^{i\mu_0}\rho\left(\frac{z + a}{1 + az} + |w_0|\right)$$

for some $0 < a < 1$ and μ_0 satisfying $w_0 = |w_0| e^{i\mu_0}$.

Proof. If $|z| = 1/3$, then it follows from (13) that

$$\sum_{n=1}^{\infty} \left|e^{i\mu} a_n + e^{-i\mu} b_n\right| \left|z^n\right| \le \rho - \left|\operatorname{Re} e^{i\mu}(a_0 - w_0)\right|,$$

and (17) is evident. On the other hand, (14) yields

$$|a_n| + |b_n| \le \frac{4}{\pi}\rho,$$

and with $|z| = 1/3$ gives

$$\sum_{n=1}^{\infty} \left|a_n z^n\right| + \sum_{n=1}^{\infty} \left|b_n z^n\right| \le \frac{1}{2}\left(\frac{4}{\pi}\rho\right) = \frac{2}{\pi}\rho,$$

which is (16).

For sharpness, consider the Möbius transformation

$$\varphi(z) = e^{i\mu_0}\rho\left(\frac{z + a}{1 + az} + |w_0|\right), \quad a > 0.$$

Then

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n = e^{i\mu_0}\rho\big(a + |w_0|\big) + e^{i\mu_0}\rho\big(1 - a^2\big) \sum_{n=1}^{\infty} (-a)^{n-1} z^n$$

yields

$$\varphi^*(z) = \sum_{n=0}^{\infty} |a_n| z^n = \rho\big(a + |w_0|\big) + \sum_{n=1}^{\infty} \rho\big(1 - a^2\big) a^{n-1} z^n = \rho\big(2a + |w_0|\big) + \rho\left(\frac{z - a}{1 - az}\right).$$

For any fixed r_0, a brief computation shows that

$$\frac{1}{\rho}\varphi^*(r_0) - |w_0| = \frac{a + (1 - 2a^2)r_0}{1 - ar_0} \geq 1$$

provided $r_0 \geq 1/(1 + 2a)$, or equivalently $a \geq (1/2)(1/r_0 - 1)$. Hence for any $r_0 > 1/3$, there exists an a satisfying $1 > a \geq (1/2)(1/r_0 - 1)$ where (17) does not hold in the open disk D with radius $\rho > 0$ centered at ρw_0. Hence the bound $1/3$ is best possible. □

Remark 4.5. In the case D is the unit disk U, Theorem 4.4 reduces to Theorem 2 in [1].

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